UNIT - III

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Maxwell's Equations.

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MAXWELL'S EQUATIONS

→ The electrostatic fields are usually produced by static electric charges. whereas magnetostatic fields are due to motion of electric charges with uniform velocity (direct current) (or) static magnetic charges.

→ The time-varying fields are (or) waves are usually due to accelerated charges (or) time-varying currents.

* Stationary charges → Electrostatic fields.
* Steady currents → Magnetostatic fields.
* Time-varying currents → Electromagnetic fields (or) waves.

→ The relationship between time varying electric and magnetic fields are known as Maxwell's equations.

Faraday's Law :

Faraday's law states that the induced electromotive force (emf) (or) V_{emf} in any closed circuit is equal to the rate of change of magnetic flux linkage (λ) by the circuit.

(or)

It states that in a magnetic field, if the conductor is in motion (or) if the field is time varying then the induced emf in a closed circuit is equal to the time rate of change of magnetic flux linkage by the circuit.

→ It can be expressed as $V_{emf} = \frac{-d\lambda}{dt} = -\frac{d}{dt}(N\psi)$ $\quad \because \lambda = N\psi$

$$\boxed{\therefore V_{emf} = -N\frac{d\psi}{dt}} \text{ volts.}$$

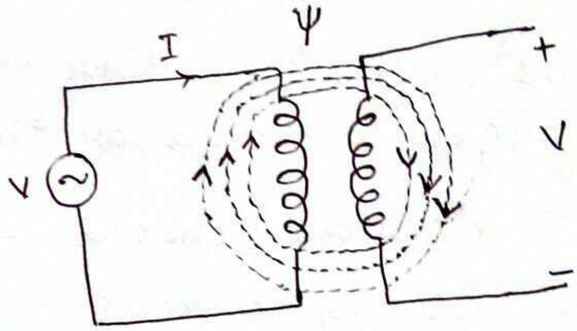
→ The negative sign indicates that the induced emf and flux are in opposit direction is called Lenz's law.

→ The e.m.f is nothing but a voltage that induces from changing magnetic fields (or) motion of conductors in a magnetic field

→ There are three conditions for the induced emf.

1. Transformer emf (or) Stationary loop in time-varying B field.
2. Generator emf (or) motional emf (or) Moving loop in Static B field.
3. Both Transformer & Generator emf (or) Moving loop in Time varying field.

① Transformer emf :



→ Consider a transformer (stationary ckt) as shown in fig.

→ When a time-varying current is applied, it produces a time varying magnetic flux in the primary coil, that induces an emf in the secondary coil. This emf is called Transformer emf (or) statically induced emf.

→ According to faraday's law, (for $N=1$)

$$V = -\frac{d\psi}{dt}$$

→ But we know that $\psi = \int_S B \cdot dS$ and $V = \oint_L E \cdot dl$

then $$\oint_L E \cdot dl = -\frac{d}{dt}\int_S B \cdot dS$$

→ Using stokes theorem on LHS side

$$\int_S (\nabla \times E) \cdot dS = -\int_S \frac{dB}{dt} \cdot dS$$

→ Removing surface integrals on both sides, then

$$\boxed{\nabla \times E = -\frac{\partial B}{\partial t}}$$

→ This is one of the maxwell's equation for time varing fields. It shows that time-varying E field is not conservative ($\nabla \times E \neq 0$).

→ If the flux density is constant for static field, then the field is conservative i.e., $\nabla \times E = 0$ and

$$\oint_L E \cdot dl = 0.$$

Equation of Continuity for time varying Fields :

→ When the current is flowing through a conductor due to conservation of charge, the total current flowing out from a volume must be equal to the rate of decrease of charg within the volume.

→ The sum of divergence of current density (J) and rate of change in volume charge density (ρ_v) is equal to zero is called continuity equation (or) equation of continuity for time varying fields.

$$\boxed{\nabla \cdot J + \frac{d\rho_v}{dt} = 0}$$

Proof:

We know that current through the closed surface is

$$I = \oint_S J \cdot dS$$

$\because J = \frac{dI}{dS}$

$dI = J \cdot dS$

$I = \int_S J \cdot dS$

Apply divergence theorem

$$I = \int_{vol} (\nabla \cdot J)\, dv \longrightarrow (1)$$

→ current flow with in the closed surface is

$$I = -\frac{dQ}{dt}$$

$$I = -\frac{d}{dt}\left[\int_{vol} \rho_v\, dv\right]$$

$$I = \int_{vol} -\frac{d\rho_v}{dt} \cdot dv \longrightarrow (2)$$

$\rho_v = \frac{dQ}{dv}$

$dQ = \rho_v\, dv$

$Q = \int \rho_v\, dv$

→ Equating eq (1) & (2)

$$\int_{vol} (\nabla \cdot J)\, dv = \int_{vol} -\frac{d\rho_v}{dt}\, dv$$

Removing volume integrals on both sides

$$\boxed{\nabla \cdot J = -\frac{d\rho_v}{dt}} \Rightarrow \boxed{\nabla \cdot J + \frac{d\rho_v}{dt} = 0}$$

Inconsistency of Ampere's law :

→ From the definition of ampere's circuit law

$$\oint_L H \cdot dl = I$$

Apply stokes theorem on L.H.S. side

$$\int_S (\nabla \times H) \cdot dS = I \qquad \Big| \because I = \int_S J \cdot dS$$

$$\int_S (\nabla \times H) \cdot dS = \int_S J \cdot dS$$

Removing Surface integrals on both sides

$$\boxed{\nabla \times H = J}$$

Apply divergence operation

$$\nabla \cdot (\nabla \times H) = \nabla \cdot J$$

→ The divergence of the curl of any vector field is identically zero.

Hence $\quad 0 = \nabla \cdot J \qquad \Big| \because \nabla \cdot (\nabla \times H) = 0$

$$\boxed{\therefore \nabla \cdot J = 0}$$

→ from continuity equation $\boxed{\nabla \cdot J = -\frac{\partial \rho_v}{dt} \neq 0}$

If $\boxed{\nabla \cdot J = 0}$ then $\boxed{\frac{\partial \rho_v}{dt} = 0}$

→ The above quantity can not be zero. That means

$\int_L H \cdot dl = I$ (or) $\nabla \times H = J$ are not compatiable

for time-varying conditions.

→ therefore ampere's circuit law is inconsistant.

Displacement Current Density :

→ We know that the ampere's law $\oint_l H \cdot dl = I$ (or) Maxwell's equation $\nabla \times H = J$ are not compatiable for time-varying conditions.

→ Therefore some modification is required for maxwell's equation. $\nabla \times H = J$. Replace current density J with $J + J_d$. Where J_d is the displacement current density & J conduction current density.

$$\therefore \nabla \times H = J + J_d \longrightarrow (1)$$

Apply divergence operation

$$\nabla \cdot (\nabla \times H) = \nabla \cdot (J + J_d)$$

$$0 = \nabla \cdot J + \nabla \cdot J_d$$

$$\therefore \nabla \cdot J_d = -\nabla \cdot J$$

$$\nabla \cdot J_d = \frac{\partial \rho_v}{dt} \qquad \because \text{Continuity eq } \nabla \cdot J = -\frac{\partial \rho_v}{dt}$$

$$\nabla \cdot J_d = \frac{\partial}{\partial t}(\nabla \cdot D) \qquad \because \text{Gauss's law } \nabla \cdot D = \rho_v.$$

$$\nabla \cdot J_d = \nabla \cdot \frac{\partial D}{\partial t}$$

$$\boxed{\therefore J_d = \frac{\partial D}{\partial t}}$$

∴ Rate of change of electric flux density D is called displacement current density (J_d).

→ from eq (1) $\nabla \times H = J + J_d \Rightarrow \boxed{\nabla \times H = J + \frac{\partial D}{\partial t}}$

→ This is the maxwell's equation based on Ampere's circuit law for time-varying field.

Displacement Current :

→ The current flow due to time-varying electric field is called displacement current. (or) The current flow through dielectric material is called displacement current.

→ It is denoted with I_D and can be computed from displacement current density

$$\boxed{I_D = \oint_S J_D \cdot dS}$$ where $J_D = \frac{\partial D}{\partial t}$

Conduction current :

→ The current flow through the conductor (or) conducting material is called conduction current

→ It is denoted with I_c and can be computed from conduction current density.

$$\boxed{I_c = \oint_S J \cdot dS}$$ where $J = \sigma E$.

Problem (9.4) A Parallel plate capacitor with plate area of $5\ cm^2$ and plate seperation of 3 mm has a voltage $50 \sin 10^3 t$ V applied to its plates. Calculate the displacement current assuming $\epsilon = 2\epsilon_0$.

Sol
plate area $S = 5\ cm^2 = 5\times10^{-4}\ m^2$
plate seperation $d = 3\ mm = 3\times10^{-3}\ m$

Voltage $V = 50 \sin 10^3 t$ V
$\epsilon = 2\epsilon_0$.

$$I_D = \oint_S J_d \cdot dS = J_d \cdot S = S \cdot \frac{\partial D}{\partial t}$$ $\because J_d = \frac{\partial D}{\partial t}$

$$I_D = S \cdot \frac{d}{dt}(\epsilon E) = S\epsilon \frac{\partial (E)}{\partial t} = S\epsilon \frac{\partial}{\partial t}\left(\frac{V}{d}\right)$$ $D = \epsilon E$, $E = \frac{V}{d}$

$$I_D = \frac{S\epsilon}{d} \cdot \frac{dV}{dt} = \frac{5\times10^{-4}\times 2\epsilon_0}{3\times10^{-3}} \cdot \frac{d}{dt}(50 \sin 10^3 t)$$

$$I_D = \frac{5\times10^{-4}\times 2}{10^9\times36\pi\times3\times10^{-3}} \; 50\cos(10^3 t)\times10^3 = 147.4 \cos 10^3 t\ nA$$

$$\boxed{\therefore \quad I_D = 147.4 \cos 10^3 t\ nA}$$

Relaxation Time :

It is the time taken by the point charge when placed in a interior of the conductor drops to e^{-1} (= 36.8%) percent of its initial value.

→ Consider point form of ohm's law (or) W.K.T $J = \sigma E$

→ from Continuity Equation $\nabla . J = -\frac{d\rho_v}{dt}$.

$$\nabla . (\sigma E) = -\frac{d\rho_v}{dt} \Rightarrow \sigma (\nabla . E) = -\frac{\partial \rho_v}{dt}$$

$$\sigma \left(\nabla . \frac{D}{\epsilon_0}\right) = -\frac{\partial \rho_v}{dt} \Rightarrow \frac{\sigma}{\epsilon_0} (\nabla . D) = -\frac{\partial \rho_v}{dt}$$

$$\frac{\sigma}{\epsilon_0} (\rho_v) = -\frac{\partial \rho_v}{dt} \Rightarrow \frac{\sigma}{\epsilon_0} dt = -\frac{d\rho_v}{\rho_v}$$

$$-\frac{\sigma}{\epsilon_0} dt = \frac{d\rho_v}{\rho_v}$$

$\int \frac{1}{x} dx = \ln x$

$c = \ln \rho_{v0}$

$$-\frac{\sigma}{\epsilon_0} t + c = \ln \rho_v$$

$$-\frac{\sigma}{\epsilon_0} t = \ln \rho_v - \ln \rho_{v0}$$

$$-\frac{\sigma}{\epsilon_0} t = \ln \left(\frac{\rho_v}{\rho_{v0}}\right)$$

$$\frac{\rho_v}{\rho_{v0}} = e^{-\frac{\sigma}{\epsilon_0} t}$$

$$\boxed{\rho_v = \rho_{v0} e^{-\frac{\sigma}{\epsilon_0} t} = \rho_{v0} e^{-t/\tau}} \quad \therefore \tau = \frac{\epsilon_0}{\sigma}$$

This expression shows that the volume charge density exponentially decays with time constant.

→ At $t = 0$; $\rho_v = \rho_{v0}$

→ At $t = \frac{\epsilon_0}{\sigma}$; $\rho_v = \rho_{v0} e^{-1} = 37\%$ of ρ_{v0}.

Maxwell's Equations for time varying Fields :

We know that the stationary charges produces electrostatic fields, steady current produces magnetostatic fields and the time-varying current produces electromagnetic fields E, D, H and B.

→ Maxwell's equations are the relations among the electric field intensity E, electric flux density D, magnetic field intensity H and magnetic flux density B.

(1). $\nabla \cdot D = \rho_v$

(2). $\nabla \cdot B = 0$

(3) $\nabla \times E = -\frac{dB}{dt}$

(4) $\nabla \times H = J + \frac{dD}{dt}$.

Equation 1 :

Divergence of electric flux density is equal to the volume charge density ρ_v

$$\boxed{\nabla \cdot D = \rho_v}$$

Proof : from the definition of Gauss's law

$$\Psi = Q = \oint_S D \cdot dS$$

Apply divergence theorem

$$Q = \int_{vol} (\nabla \cdot D)\, dv \longrightarrow (1)$$

The charge enclosed by the volume is

$$Q = \int_{vol} \rho_v\, dv \longrightarrow (2)$$

Equating eq (1) & (2)

$$\boxed{\int_{vol} (\nabla \cdot D)\, dv = \int_{vol} \rho_v\, dv} \rightarrow \text{It is integral form}$$

Removing volume integrals on both sides

$$\boxed{\nabla \cdot D = \rho_v} \rightarrow \text{It is Point form (or) differential form.}$$

Equation 2 :

Divergence of magnetic flux density B around a closed loop is zero.

$$\boxed{\nabla \cdot B = 0}$$

Proof: The total flux through any closed surface with in the magnetic field B is zero

$$\psi = 0$$

$\because \psi = \int_s B \cdot ds$

$$\boxed{\int_s B \cdot ds = 0} \rightarrow \text{It is integral form}$$

Apply divergence theorem

$$\int_{vol} (\nabla \cdot B)\, dv = 0$$

$$\boxed{\nabla \cdot B = 0} \rightarrow \text{It is Point form (or) differential form.}$$

Equation 3 :

Curl of a electric field intensity E is equal to the rate of decrease in magnetic flux density B.

$$\boxed{\nabla \times E = -\frac{\partial B}{\partial t}}$$

Proof: from faradays law of electromagnetic induction.

$$V_{emf} = \frac{-d\lambda}{dt} = \frac{-d}{dt} N\psi$$

$\because \lambda = N\psi$

$V_{emf} = -V = \oint_L E \cdot dl$

$$V_{emf} = -N\frac{d\psi}{dt} = -\frac{d\psi}{dt} \quad (\text{For } N=1)$$

But we know that $V_{emf} = \oint_L E \cdot dl$ and $\psi = \int_s B \cdot ds$

then $$\oint_L E \cdot dl = -\frac{d}{dt}\int_s B \cdot ds$$

Apply stokes theorem to the LHS

$$\int_S (\nabla \times E) \cdot ds = -\int_S \frac{dB}{dt} \cdot ds$$

Removing Surface integrals on both sides

$$\boxed{\nabla \times E = \frac{-dB}{dt}}$$ → It is Point form (or) differential form.

$$\boxed{\oint_L E \cdot dl = -\int_S \frac{dB}{dt} \cdot ds}$$ → It is Integral form.

Equation 4 :

Curl of magnetic field intensity H is equal to the sum of conduction and displacement current densities.

$$\boxed{\nabla \times H = J + \frac{dD}{dt}}$$

Proof :

We know that the amperes law $\oint_L H \cdot dl = I$ and maxwell's equation $\nabla \times H = J$ are not compatiable for time varying conditions.

→ Therefore Some modification is required for maxwell's equation $\nabla \times H = J$

Replace current density J with $J + J_d$

$$\nabla \times H = J + J_d \longrightarrow (1)$$

Apply divergence operation

$$\nabla \cdot (\nabla \times H) = \nabla \cdot (J + J_d)$$

$$0 = \nabla \cdot J + \nabla \cdot J_d$$

$$\therefore \nabla \cdot J_d = -\nabla \cdot J = \frac{d \rho_v}{dt} = \frac{d}{dt}(\nabla \cdot D)$$

$$\nabla \cdot J_d = \nabla \cdot \frac{dD}{dt}$$

$$J_d = \frac{dD}{dt}$$

Continuity eq

$$\nabla \cdot J = \frac{-d\rho_v}{dt}$$

Gauss law

$$\nabla \cdot D = \rho_v$$

∴ from eq (1) $$\boxed{\nabla \times H = J + \frac{dD}{dt}}$$ → It is Point form (or) differential form

$$\boxed{\oint_L H \cdot dl = \int_S \left(J + \frac{dD}{dt}\right) \cdot ds}$$ → It is integral form.

Maxwell's equations in final forms :

Point (differential) form	Integral form	Remarks
(1) $\nabla \cdot D = \rho_v$	$\oint_S D \cdot dS = \int_{vol} \rho_v \, dv$	Gauss's law for electrostatic field
(2) $\nabla \cdot B = 0$	$\oint_S B \cdot dS = 0$	Gauss's law for magneto static field
(3) $\nabla \times E = \frac{-\partial B}{\partial t}$	$\oint_L E \cdot dl = -\int_S \frac{dB}{dt} \cdot dS$	faraday's law
(4) $\nabla \times H = J + \frac{\partial D}{\partial t}$	$\oint_L H \cdot dl = \int_S \left(J + \frac{\partial D}{\partial t}\right) \cdot dS$	Ampere's circuit law.

Maxwell's equations for free space : $(\sigma = 0, J = 0 \; \& \; \rho_v = 0)$.

Point (or) Differential form	Integral form
(1) $\nabla \cdot D = 0$	$\oint_S D \cdot dS = 0$
(2) $\nabla \cdot B = 0$	$\oint_S B \cdot dS = 0$
(3) $\nabla \times E = \frac{-\partial B}{\partial t}$	$\oint_L E \cdot dl = \int_S \frac{\partial B}{\partial t} \cdot dS$
(4) $\nabla \times H = \frac{\partial D}{\partial t}$	$\oint_L H \cdot dl = \int_S \frac{\partial D}{\partial t} \cdot dS$

Practice Ex 9.4 In free space, $E = 20\cos(\omega t - 50x)\,a_y$ V/m

Calculate a) J_d b) H c) ω.

Sol

In free space $\sigma = 0$; $J = \sigma E = 0$ and $\rho_v = 0$.

$\epsilon = \epsilon_0$ and $\mu = \mu_0$

Given $E = 20\cos(\omega t - 50x)\,a_y$ v/m.

a) Displacement current density (J_d):

$$J_d = \frac{dD}{dt} = \frac{d(\epsilon_0 E)}{dt} = \epsilon_0 \frac{dE}{dt}$$

$$J_d = \epsilon_0 \frac{d}{dt}\left(20\cos[\omega t - 50x]\,a_y\right)$$

$$J_d = \epsilon_0\, 20\left[-\sin(\omega t - 50x)\,a_y\right].\omega$$

$$\boxed{J_d = -20\omega\epsilon_0 \sin(\omega t - 50x)\,a_y \text{ Amp/m}^2}$$

b) Magnetic field Intensity (H):

from Maxwell's 3rd equation $\nabla \times E = -\frac{dB}{dt}$

then $\frac{dB}{dt} = -\nabla \times E$

$$\therefore \nabla \times E = \begin{vmatrix} a_x & a_y & a_z \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ 0 & 20\cos(\omega t - 50x) & 0 \end{vmatrix}$$

$$\nabla \times E = a_x\left[0 - \frac{d}{dz}(20\cos[\omega t - 50x])\right] - a_y[0-0] + a_z\left[\frac{d}{dx}(20\cos[\omega t - 50x]) - 0\right]$$

$$\nabla \times E = a_x(0) - a_y(0) + a_z\left[-20\sin(\omega t - 50x)\cdot(-50)\right]$$

$$\nabla \times E = 1000\sin(\omega t - 50x)\,a_z.$$

$$\therefore \frac{dB}{dt} = -\nabla \times E = -1000\sin(\omega t - 50x)\,a_z$$

Integrate on both sides w.r.t 't'

then $B = -1000 \int \sin(\omega t - 50x)\, a_z \cdot dt$

$$B = (-1000) \frac{-\cos(\omega t - 50x)\, a_z}{\omega} = \frac{1000 \cos(\omega t - 50x)\, a_z}{\omega}$$

But we know that $B = \mu_0 H \Rightarrow H = \frac{B}{\mu_0}$

then $$\boxed{H = \frac{1000 \cos(\omega t - 50x)\, a_z}{\omega \mu_0}}$$

c) frequency (ω):

from 4th maxwell's equation $\nabla \times H = J + J_d = J_d$ $\quad |\because J = 0$

$$\nabla \times H = \begin{vmatrix} a_x & a_y & a_z \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ 0 & 0 & \frac{1000}{\omega \mu_0} \cos(\omega t - 50x) \end{vmatrix}$$

$$\nabla \times H = a_x \left[\frac{d}{dy} \left(\frac{1000}{\omega \mu_0} \cos(\omega t - 50x) - 0 \right] - a_y \left[\frac{d}{dx} \left(\frac{1000}{\omega \mu_0} \cos(\omega t - 50x) \right) - 0 \right] + a_z [0 - 0]$$

$$\nabla \times H = a_x [0] - a_y \left[\frac{1000}{\omega \mu_0} (-\sin(\omega t - 50x) \cdot (-50)) \right] + a_z [0]$$

$$\nabla \times H = -\frac{50000}{\omega \mu_0} \sin(\omega t - 50x]\, a_y$$

But we know that in free space $\nabla \times H = J_d$

ie $$-\frac{50000}{\omega \mu_0} \sin(\omega t - 50x)\, a_y = -20 \omega \epsilon_0 \sin(\omega t - 50x)\, a_y$$

$$\frac{\cancel{50000}^{2500}}{\omega \mu_0} = \cancel{20}\, \omega \epsilon_0$$

$$\omega^2 = \frac{2500}{\mu_0 \epsilon_0} = \frac{2500 \times 10^9 \times \cancel{36\pi}^{9}}{4\pi \times 10^{-7}}$$

$\epsilon_0 = \frac{1}{10^9 \times 36\pi}$

$\mu_0 = 4\pi \times 10^{-7}$

$$\omega^2 = 2500 \times 9 \times 10^9 \times 10^7$$

$$\omega^2 = 22500 \times 10^{16}$$

$$\omega = \sqrt{22500 \times 10^{16}}$$

$$\omega = 150 \times 10^8 \text{ rad/sec}$$

$$\boxed{\omega = 1.5 \times 10^{10} \text{ rad/sec}}$$

Problem (9.8) In medium characterised by $\sigma = 0$, $\mu = \mu_0$, $\epsilon = \epsilon_0$ and $E = 20 \sin(10^8 t - \beta z)\, a_y$ v/m calculate a) β and b) H.

Sol

Given $E = 20 \sin(10^8 t - \beta z)\, a_y$ v/m

from Maxwell's 3rd equation

$$\nabla \times E = -\frac{dB}{dt} \longrightarrow (1)$$

$$\therefore \nabla \times E = \begin{vmatrix} a_x & a_y & a_z \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ 0 & 20\sin(10^8 t - \beta z) & 0 \end{vmatrix} = a_x\left[0 - \frac{d}{dz}\left(20 \sin(10^8 t - \beta z)\right)\right] - a_y[0-0] + a_z\left[\frac{d}{dx}\left(20 \sin(10^8 t - \beta z)\right) - 0\right]$$

$$\nabla \times E = -a_x\left[20 \cos(10^8 t - \beta z)(-\beta)\right] - a_y(0) + a_z[0]$$

$$\nabla \times E = +20\beta \cos(10^8 t - \beta z)\, a_x$$

from (1) $\nabla \times E = -\frac{d}{dt}(\mu_0 H) = -\mu_0 \frac{d(H)}{dt} \Rightarrow \frac{d(H)}{dt} = \frac{-1}{\mu_0}(\nabla \times E)$

$$H = \frac{-1}{\mu_0} \int (\nabla \times E)\, dt$$

$$\therefore H = \frac{-1}{\mu_0} \int +20\beta \cos(10^8 t - \beta z)\, a_x \, dt$$

$$H = \frac{-20\beta}{\mu_0} \frac{\sin(10^8 t - \beta z)\, a_x}{10^8} = \frac{-20\beta}{\mu_0 \times 10^8} \sin(10^8 t - \beta z)\, a_x$$

$$\nabla \times H = \begin{vmatrix} a_x & a_y & a_z \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ \frac{-20\beta}{\mu_0 \times 10^8}\sin(10^8 t - \beta z) & 0 & 0 \end{vmatrix} = a_x[0-0] - a_y\left[0 - \frac{d}{dz}\left(\frac{-20\beta}{10^8 \mu_0}\sin(10^8 t - \beta z)\right)\right] + a_z\left[0 - \frac{d}{dy}\left(\frac{-20\beta}{10^8 \mu_0}\sin(10^8 t - \beta z)\right)\right]$$

$$\nabla \times H = a_x(0) - a_y\left[\frac{20\beta}{10^8 \mu_0}\cos(10^8 t - \beta z)(-\beta)\right] + a_z[0]$$

$$\nabla \times H = \frac{20\beta^2}{10^8 \mu_0} \cos(10^8 t - \beta z)\, a_y$$

$\nabla \times H = J + J_d$ If $\sigma = 0$ then $J = \sigma E = 0$

$$\nabla \times H = J_d = \frac{dD}{dt} = \epsilon_0 \frac{\partial E}{dt} \Rightarrow \frac{\partial E}{dt} = \frac{1}{\epsilon_0} \nabla \times H$$

$$\Rightarrow E = \frac{1}{\epsilon_0} \int (\nabla \times H)\, dt$$

$$\therefore E = \frac{1}{\epsilon_0} \int \frac{20\beta^2}{10^8 \mu_0} \cos(10^8 t - \beta z) a_y\, dt$$

$$E = \frac{\frac{20\beta^2}{10^8 \mu_0 \epsilon_0} \sin(10^8 t - \beta z) a_y}{10^8} = \frac{20\beta^2}{10^{16} \mu_0 \epsilon_0} \sin(10^8 t - \beta z) a_y$$

Now comparing this E with given E

$$\frac{\cancel{20}\beta^2}{10^{16}\mu_0\epsilon_0} \sin(10^8 \cancel{t - \beta z}) a_y = \cancel{20} \sin(10^8 \cancel{t - \beta z}) a_y$$

$$\beta^2 = 10^{16}\mu_0\epsilon_0 \Rightarrow \beta = 10^8 \sqrt{\mu_0 \epsilon_0} = 10^8 \sqrt{\frac{4\pi \times 10^{-7}}{36\pi \times 10^9}}$$

$$\beta = 10^8 \sqrt{\frac{10^{-16}}{9}} = 10^8 \times 10^{-8} \times \frac{1}{3} = \frac{1}{3} = 0.33$$

$$\boxed{\therefore \beta = \pm 0.33} \quad \text{(or)} \quad \boxed{\beta = \pm \frac{1}{3}}$$

then $H = \frac{-1}{6\pi} \sin(10^8 t - \frac{1}{3} z) a_x$ for $\beta = +\frac{1}{3}$

$H = \frac{1}{6\pi} \sin(10^8 t + \frac{1}{3} z) a_x$ for $\beta = -\frac{1}{3}$

Problem Ex 9.8 A medium is characterised by $\sigma = 0$, $\mu = 2\mu_0$ and $\epsilon = 5\epsilon_0$. If $H = 2\cos(\omega t - 3y) a_z$ A/m. Calculate a) ω and b) E

Sol

From maxwell's 4th Equation $\nabla \times H = J + J_d = 0 + \frac{dD}{dt}$

$$\nabla \times H = \frac{d\epsilon E}{dt} = \epsilon \frac{dE}{dt} \Rightarrow \frac{dE}{dt} = \frac{1}{\epsilon} \nabla \times H$$

$$\Rightarrow E = \frac{1}{5\epsilon_0} \int \nabla \times H \cdot dt \longrightarrow (1)$$

$$\nabla \times H = \begin{vmatrix} a_x & a_y & a_z \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ 0 & 0 & 2\cos(\omega t - 3y) \end{vmatrix} = a_x\left[\frac{d}{dy}(2\cos(\omega t - 3y)) - 0\right] - a_y[0-0] + a_z[0-0]$$

$$= 2(-\sin(\omega t - 3y)(-3) a_x$$

$$\nabla \times H = 6\sin(\omega t - 3y) a_x$$

from eq (1)

$$E = \frac{1}{5\epsilon_0}\int(\nabla\times H)\,dt = \frac{1}{5\epsilon_0}\int 6\sin(\omega t - 3y)\,a_x$$

$$E = \frac{6}{5\epsilon_0}\left(\frac{-\cos(\omega t-3y)\,a_x}{\omega}\right) = \frac{-6}{5\omega\epsilon_0}\cos(\omega t - 3y)\,a_x \longrightarrow (2)$$

from maxwell's 3rd equation

$$\nabla\times E = \frac{-dB}{dt} = \frac{-d\mu H}{dt} = -\mu\frac{d(H)}{dt}$$

$$\Rightarrow \frac{d(H)}{dt} = \frac{-1}{\mu}\nabla\times E \Rightarrow H = \frac{-1}{2\mu_0}\int(\nabla\times E)\,dt \longrightarrow (3)$$

$$\nabla\times E = \begin{vmatrix} a_x & a_y & a_z \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ \frac{-6}{5\omega\epsilon_0}\cos(\omega t-3y) & 0 & 0 \end{vmatrix} = a_x[0-0] - a_y[0-0] + a_z\left[0 - \frac{d}{dy}\left(\frac{-6}{5\omega\epsilon_0}\cos(\omega t - 3y)\right)\right]$$

$$\nabla\times E = a_x(0) + a_y(0) + a_z\left[\frac{6}{5\omega\epsilon_0}\left[-\sin(\omega t-3y)(-3)\right]\right]$$

$$\nabla\times E = \frac{18}{5\omega\epsilon_0}\sin(\omega t - 3y)\,a_z$$

from eq(3) $H = \frac{-1}{2\mu_0}\int\frac{18}{5\omega\epsilon_0}\sin(\omega t-3y)\,a_z\,dt$

$$H = \frac{-18}{10\omega\epsilon_0\mu_0}\,\frac{-\cos(\omega t-3y)\,a_z}{\omega} = \frac{18}{10\omega^2\epsilon_0\mu_0}\cos(\omega t-3y)\,a_z$$

Now compare this H with given H

$$\frac{\cancel{18}\,9}{10\omega^2\epsilon_0\mu_0}\cos(\omega t - 3y)\,a_z = \cancel{2}\cos(\omega t-3y)\,a_z$$

$$\omega^2 = \frac{9}{10\,\epsilon_0\mu_0} = \frac{9\times 36\pi\times 10^9}{10\times 4\pi\times 10^{-7}} = \frac{81\times 10^9\times 10^7}{10}$$

$$\omega^2 = \frac{81}{10}\times 10^{16} = 8.1\times 10^{16}$$

$$\boxed{\omega = 2.846\times 10^8 \text{ rad/sec}}$$

Substitute ω in eq (2)

$$E = \frac{-6}{5\omega\epsilon_0} \cos(\omega t - 3y)\, a_x$$

$$\boxed{E = -476.8 \cos(2.846 \times 10^8 t - 3y)\, a_x \ \text{V/m}}$$

Electric Boundary conditions :

When an electric field passes from one medium to other medium, it is important to study the conditions at the boundary between the two media.

Def: The conditions existing at the boundary of the two media when field passes from one medium to other medium are called boundary conditions.

→ Depending upon the nature of the media, there are two situations of the boundary conditions.

(1) Boundary b/w two perfect dielectrics

(2) Boundary b/w conductor & dielectric (or) conductor & free space.

→ For studying the boundary conditions, the maxwell's equations for electrostatics are required.

$$\boxed{\oint_L E \cdot dl = 0} \quad \text{and} \quad \boxed{\oint_S D \cdot ds = Q.}$$

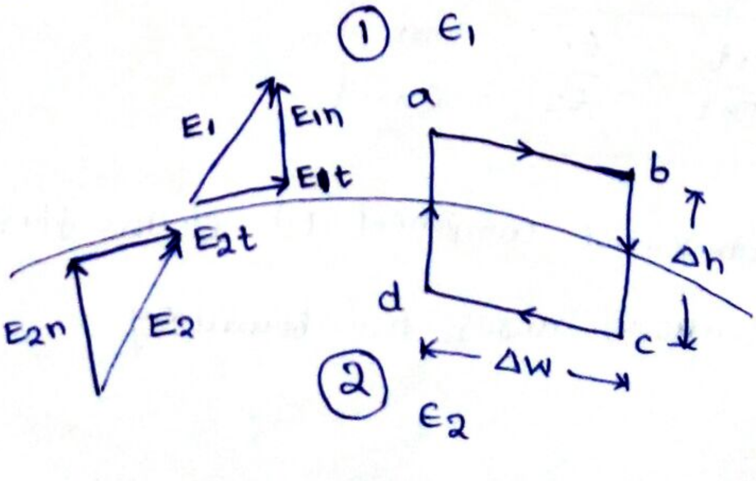
Similarly E & D are decomposed into two components.

$$\boxed{\begin{aligned} E &= E_t + E_n \\ D &= D_t + D_n \end{aligned}}$$

t → tangential to the boundary
n → normal to the boundary.

(1) Boundary conditions b/w Two Perfect Dielectrics :

let us consider one dielectric has permitivity ϵ_1 while other has permitivity ϵ_2. The interface is shown in below fig.



E_t at Boundary.

Consider a closed path abcda rectangular in shape having elementary height Δh & elementary width Δw, as shown in above fig. Let us evaluate integral of $E.dl$ along this path in clockwise direction as a.b.c.d.a.

$$\oint E.dl = 0$$

$$\therefore \int_a^b E.dl + \int_b^c E.dl + \int_c^d E.dl + \int_d^a E.dl = 0$$

Now the rectangle to be reduced at the surface to analyse the boundary conditions. Then $\Delta h \to 0$.

As $\Delta h \to 0$, $\int_b^c$ & $\int_d^a$ becomes zero.

therefore $\int_a^b E.dl + \int_c^d E.dl = 0$

$$E_{1t}\,\Delta w + (-E_{2t}\,\Delta w) = 0$$

$$E_{1t}\,\Delta W = E_{2t}\,\Delta W$$

$$\boxed{E_{1t} = E_{2t}}$$

Thus the tangential components of electric field intensity at the boundary in both the dielectrics remain same. i.e electric field intensity is continuous across the boundary.

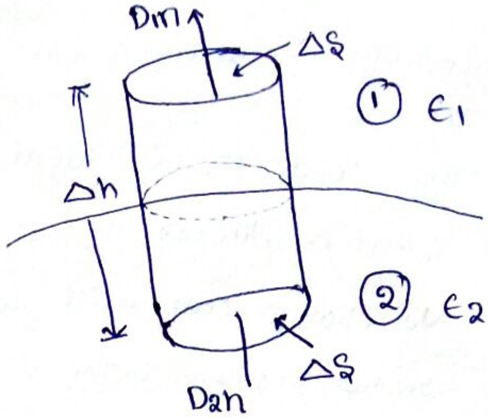
→ The relation b/w D & E is $D = \epsilon E \Rightarrow E = \frac{D}{\epsilon}$.

$$\therefore \frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}$$

$$\boxed{\frac{D_{1t}}{D_{2t}} = \frac{\epsilon_1}{\epsilon_2} = \frac{\epsilon_{r1}}{\epsilon_{r2}}}$$

Thus the tangential component of electric flux density D is discontinuous across the boundary.

D_n at Boundary



To find normal components,

let us, use Gauss's law.

→ consider a Gaussian Surface of right circular cylinder as show in fig.

According to Gauss's law, $\oint_S D \cdot dS = Q$.

$$\therefore \int_{top} D \cdot dS + \int_{bottom} D \cdot dS + \int_{lateral} D \cdot dS = Q$$

Now the cylinder to be reduced at the surface to analyse the boundary conditions. Then $\Delta h \to 0$

As $\Delta h \to 0$, $\int_{lateral} D \cdot dS$ becomes zero

Lateral Surface $2\pi r \Delta h$

→ $\rho_s = \frac{dQ}{dS} \Rightarrow dQ = \rho_s dS$

$Q = \int \rho_s dS = \rho_s \int dS$

$Q = \rho_s \Delta S$.

→ In a Perfect dielectric $\rho_s = 0$

therefore $\int_{top} D \cdot dS + \int_{bottom} D \cdot dS = Q$

$$\Delta S \, D_{1n} - D_{2n} \Delta S = \rho_s \Delta S$$

$$D_{1n} - D_{2n} = \rho_s$$

$$D_{1n} - D_{2n} = 0$$

$$\boxed{\therefore D_{1n} = D_{2n}}$$

Thus normal component of electric flux density D is continuous across the boundary.

→ Relation b/w D & E = $D = \epsilon E$.

$$E_{1n} \epsilon_1 = E_{2n} \epsilon_2$$

$$\boxed{\frac{E_{1n}}{E_{2n}} = \frac{\epsilon_2}{\epsilon_1} = \frac{\epsilon_{r2}}{\epsilon_{r1}}}$$

Thus normal component of Electric field intensity E is discontinous across the boundary.

(2) Boundary Conditions b/w Conductor & Dielectric:

$J = \sigma E$

$\sigma = \frac{J}{E}$

The conductor is ideal having infinite conductivity.

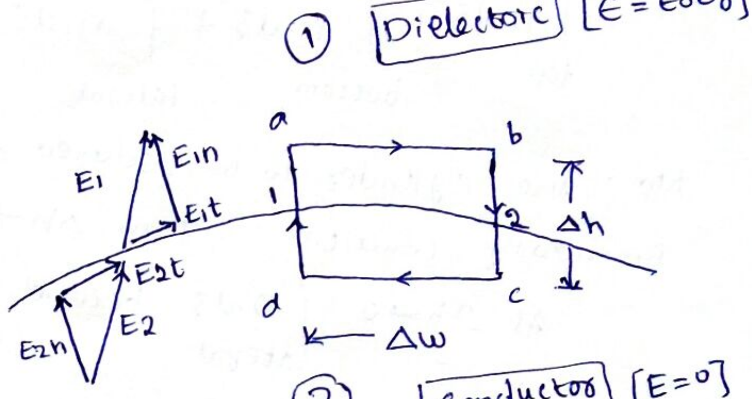
For ideal conductors (i) $E = 0$, hence $D = \epsilon E = 0$. (inside the conductor)

(ii) No charge can exist within a conductor. The charge appears on the surface in the form of surface charge density.

(iii) the charge density within the conductor is zero.

Thus E, D & ρ_v within the conductor are zero. While ρ_s is exist.

Et at the Boundary.



Consider a closed path abcda rectangular in shape having elementary hight Δh & elementary width Δw as shown in fig.

→ From Maxwell's equation

$$\oint_L E \cdot dl = 0$$

$$\therefore \int_a^b E \cdot dl + \int_b^c E \cdot dl + \int_c^d E \cdot dl + \int_d^a E \cdot dl = 0$$

$$\int_a^b E \cdot dl + \int_b^2 E \cdot dl + \int_2^c E \cdot dl + \int_c^d E \cdot dl + \int_d^1 E \cdot dl + \int_1^a E \cdot dl = 0$$

→ In a conductor $E = 0$

Hence $\int_2^c$, $\int_c^d$ and $\int_d^1$ becomes zero.

therefore

$$\int_a^b E \cdot dl + \int_b^2 E \cdot dl + \int_1^a E \cdot dl = 0.$$

$$E_{1t} \Delta w + E_{1n}\left(\frac{\Delta h}{2}\right) - E_{1n}\left(\frac{\Delta h}{2}\right) = 0$$

$$E_{1t} \Delta w = 0$$

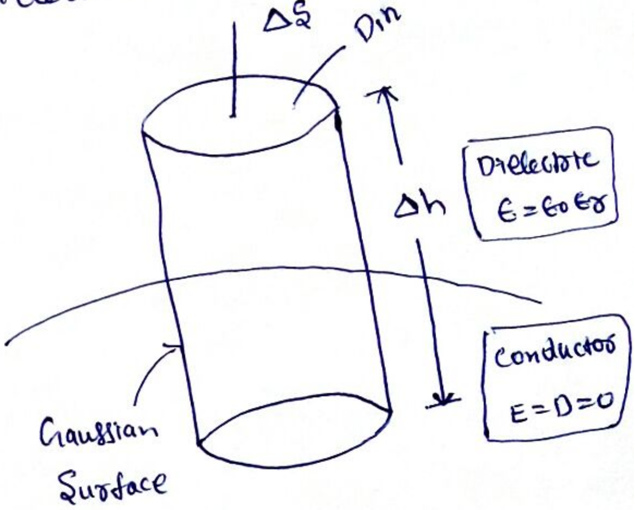
$$\boxed{E_{1t} = 0}$$

Thus the tangential component of the electric field intensity is zero at the boundary b/w conductor and dielectric.

→ Relation b/w D & E is $D = \epsilon E \Rightarrow E = \frac{D}{\epsilon}$.

$$\frac{D_{1t}}{\epsilon_1} = 0 \Rightarrow \boxed{D_{1t} = 0}$$

Thus the tangential component of electric flux density is zero at the boundary b/w conductor & dielectric.



Dn at the boundary.

To find normal components,

let us use Gauss's law.

→ consider a Gaussian Surface of right circular cylinder as shown in fig.

→ According to Gauss Law.

$$\oint_S D \cdot dS = Q$$

$$\therefore \int_{top} D \cdot dS + \int_{bottom} D \cdot dS + \int_{lateral} D \cdot dS = Q$$

→ In a conductor $D = 0$. Hence $\int_{bottom}$ becomes zero.

At the boundary $\Delta h \to 0$, ~~the~~ Hence $\int_{lateral}$ becomes zero.

And $\rho_s = \frac{dQ}{dS} \Rightarrow dQ = \rho_s \, dS \Rightarrow Q = \int_S \rho_s \, dS = \rho_s \Delta S$

$$\therefore \int_{top} D \cdot dS + 0 + 0 = Q$$

$$D_{1n} \Delta S = \rho_s \Delta S$$

$$\boxed{D_{1n} = \rho_s}$$

Thus normal component of electric flux density D is equal to the surface charge density.

Relation b/w D & E 4 $D = \epsilon E$

$E_{in}\, \epsilon = \rho_s$

$$\boxed{E_{in} = \frac{\rho_s}{\epsilon}}$$

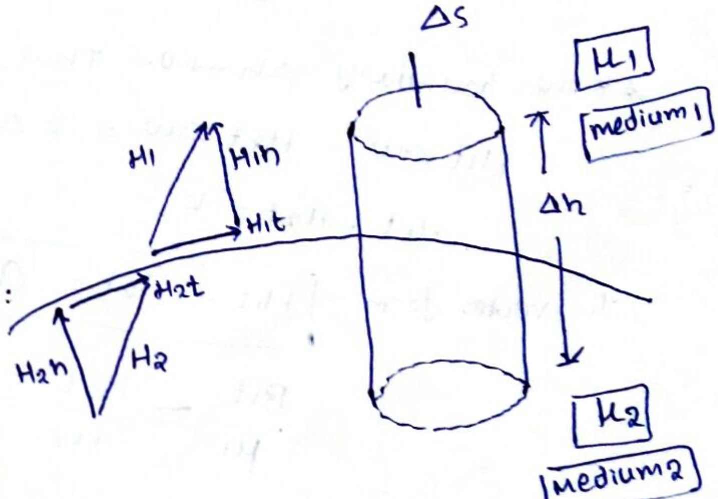
Magnetic Boundary conditions:

The conditions of the magnetic field existing at the boundary of the two media when the magnetic field passes from one medium to other medium are called magnetic boundary conditions.

→ For studying the boundary conditions, the maxwell's equations for magnetostatics are required.

$$\boxed{\oint_S B \cdot ds = 0} \quad \text{and} \quad \boxed{\oint_L H \cdot dl = I}$$

Bn at Boundary:



To find normal components, let us use Gauss's law.

→ Consider a Gaussian surface in the form of right circular cylinder as shown in fig.

→ According to Gauss law $\oint_S B \cdot ds = 0$

$$\therefore \oint_{top} B \cdot ds + \oint_{bottom} B \cdot ds + \oint_{lateral} B \cdot ds = 0$$

Now the cylinder to be reduced at the surface to analyse the boundary conditions. Then $\Delta h \to 0$; As $\Delta h \to 0$, $\int_{lateral} B \cdot ds$ becomes zero.

therefor $\oint_{top} B \cdot ds + \oint_{bottom} B \cdot ds = 0$

$$B_{1n} \Delta S - B_{2n} \Delta S = 0$$

$$\boxed{B_{1n} = B_{2n}}$$

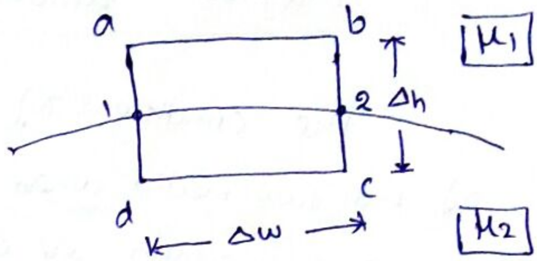
Thus normal component of B is continuous at the boundary.

→ The relation b/w B & H is $B = \mu H$

$$\mu_1 H_{1n} = \mu_2 H_{2n} \Rightarrow \boxed{\frac{H_{1n}}{H_{2n}} = \frac{\mu_2}{\mu_1} = \frac{\mu_{r2}}{\mu_{r1}}}$$

Thus normal component of H is discontinuous at the boundary.

Et at Boundary:



Consider a closed path abcda as shown in fig.

According to Amperes ckt law

$$\oint_L H \cdot dl = I$$

$$\therefore \int_a^b H \cdot dl + \int_b^2 H \cdot dl + \int_2^c H \cdot dl + \int_c^d H \cdot dl + \int_d^1 H \cdot dl + \int_1^a H \cdot dl = I$$

Assume that K is the surface current normal to the path.

$$H_{1t}\,\Delta w + H_{1n}\frac{\Delta h}{2} + H_{2n}\frac{\Delta h}{2} - H_{2t}\Delta w - H_{2n}\frac{\Delta h}{2} - H_{1n}\frac{\Delta h}{2} = K \cdot \Delta w$$

At the boundary $\Delta h \to 0$. Thus,

$$H_{1t}\,\Delta w - H_{2t}\,\Delta w = K\,\Delta w$$

$$H_{1t} - H_{2t} = K$$

In vector form $\boxed{H_{1t} - H_{2t} = a_{n12} \times K}$ $\qquad$ $a_{n12} \to$ unit vector.

$$\frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = K$$

Special case:

media are not conductors; so $K = 0$. then

$$H_{1t} - H_{2t} = 0 \Rightarrow \boxed{H_{1t} = H_{2t}}$$

Thus tangential components of H is continuous at the boundary.

$$\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2} \Rightarrow \boxed{\frac{B_{1t}}{B_{2t}} = \frac{\mu_1}{\mu_2} = \frac{\mu_{r1}}{\mu_{r2}}}$$

Thus tangential components of B is discontinuous at the boundary.